Lecture 5

03 March 2020 16:37

DIFFERENTIAL OPERATORS II

(references: Ch. 3, 9 and 11 of Nestruev - Smooth Manifolds and Observables)

Definition Proposition Exercise

I DIFFERENTIAL OPERATORS ON MANIFOLDS

I.1 From Algebras to Smooth Manifolds

Following from the algebraic theory of differential operators seen in Lecture 4, we restrict to a special class of algebras. Algebras $\mathcal{A}$ are assumed to be over the reals $\mathbb{R}$, unital, associative and commutative. We define point as an algebra morphism $x: \mathcal{A} \to \mathbb{R}$ and the dual of the algebra $|\mathcal{A}|$ as the collection of all such maps. An algebra $\mathcal{A}$ is geometric when

$$\bigcap_{x \in |\mathcal{A}|} \operatorname{Ker} x = 0$$

Defining the $\mathbb{R}$-valued functions on the dual of the form

$$\tilde{\mathcal{A}} = \{ f_a : |\mathcal{A}| \to \mathbb{R} \mid f_a(x) = x(a) \ \forall a \in \mathcal{A} \}$$

we find a canonical algebra structure on $\tilde{\mathcal{A}}$ and an isomorphism of algebras (using the fact that $\mathcal{A}$ is geometric):

$$\mathcal{A} \cong \tilde{\mathcal{A}}.$$

thus justifying the name "geometric" since all such algebras are canonically isomorphic to algebras of $\mathbb{R}$-valued functions on sets. It is possible to put a topology or, further, a smooth structure on $|\mathcal{A}|$ so that we can define the notion of continuous or smooth algebras and find an equivalence with topological spaces and smooth manifolds (see Ch. 3 of Nestruev for details).

Differential operators on a smooth manifold are defined as:

$$\operatorname{Diff}(M) = \bigcup_{k \in \mathbb{N}} \operatorname{Diff}_k(M) \quad , \operatorname{Diff}_k(M) = D^p(C^\infty(M))$$

where we have used the notation introduced in lecture 4.

II.2 Jet Bundles of Manifolds

Since differential operators were shown to correspond to the classical notion from linear PDEs locally, we are compelled to define the following equivalence relation on functions:

$$f,g \in C^\infty(M) \qquad f \sim^k_x g \Leftrightarrow f(x) = g(x)\ ,\ \partial^{|k|}_{\dots} f(x) = \partial^{|k|}_{\dots} g$$

Similarly to the construction of the cotangent bundle we define the k-jet bundle of a manifold M as:

$$J^k M := \bigcup_{x\in M} J^k_x M \quad , \quad J^k_x M = \left\{ [f]^k_x =: j^k_x f \ ,\ f \in C^\infty(M) \right\}$$

Note that, by construction we have the following maps:

$$\pi^k : J^k M \to J^{k-1} M \qquad \text{and} \qquad j^k : C^\infty(M) \to \Gamma(J^k M)$$

where π^k simply projects to classes of functions that agree on lower order derivatives and the k-jet prolongation or simply the k-jet of a function f is defined as

$$j^k f(x) = j^k_x f$$

Jet bundles are the objects that allow us to see that differential operators are, in fact, the sections of some vector bundle, as seen in the next proposition:

Proposition 5.1 $\mathrm{Diff}_k(M) \cong \Gamma(J^k M^*)$

Proof: (see section 11.47 of Nestruev for details)

the basic idea of this proof is that the jet prolongation of any function behaves as a differential operator by construction and thus any differential operator, i.e. an $\mathbb{R}$-linear map $\Delta : C^\infty(M) \to C^\infty(M)$, can be equivalently defined by a $C^\infty(M)$-linear map $\tilde{\delta} : \Gamma(J^k M) \to C^\infty(M)$ so that we set

$$\Delta = \tilde{\delta} \circ j^k .$$

Since $\tilde{\delta}$ is $C^\infty(M)$-linear, it is equivalent to a bundle map covering the identity $\delta : J^k M \to \mathbb{R}_M$ which indeed corresponds to a section of the dual jet bundle $\delta \in \Gamma(J^k M^*)$. Then the isomorphism is realised via the factorisation formula:

section of the whole jet ... isomorphism is realised via the factorisation formula:

$$\Delta = \delta_* \circ j^K .$$

//

II DIFFERENTIAL OPERATORS ON VECTOR BUNDLES

II.1 Differential Operators on Modules

Let $\mathcal{A}$ be a K-algebra and $\mathcal{P}$, $\mathcal{Q}$ A-modules. Let us introduce the following notation for the commutator of k-linear morphisms

$$a \in \mathcal{A} \quad , \quad \phi \in \mathrm{Hom}_K(\mathcal{P}, \mathcal{Q}), \; c_a(\phi) := [\phi, a] := \phi \circ \mu_{\mathcal{P}}(a) - \mu_{\mathcal{Q}}(a) \circ \phi$$

Differential operators of order $\leq K$ between the A-modules $\mathcal{P}$ and $\mathcal{Q}$ are defined as follows:

$$D^K(\mathcal{P}, \mathcal{Q}) := \left\{ \Delta \in \mathrm{Hom}_K(\mathcal{P}, \mathcal{Q}) \mid c_{a_0} \circ c_{a_1} \circ \cdots \circ c_{a_K}(\Delta) = 0 \;\; \forall a_i \in \mathcal{A} \right\}$$

Which carries a natural A-bimodule structure.

II.2 From Modules to Smooth Vector Bundles

In a similar fashion to the identification of geometric, continuous and smooth algebras with sets, topological spaces and manifolds seen in section I.1, there is an equivalence between smooth vector bundles and finitely generated projective modules over smooth algebras.

Differential operators between two vector bundles over the same manifold $A, B \to M$ are defined as.

$$\mathrm{Diff}(A,B) := \bigcup_{K \in \mathbb{N}} \mathrm{Diff}_K(A,B) \quad , \; \mathrm{Diff}_K(A,B) := D^K(\Gamma(A), \Gamma(B))$$

We will use the notation $\mathrm{Diff}_K(A) := \mathrm{Diff}_K(A,A)$. The filtered nature of differential operators gives the symbol short exact sequence:

$$0 \to \mathrm{Diff}_{K-1}(A,B) \hookrightarrow \mathrm{Diff}_K(A,B) \xrightarrow{\sigma} \Gamma(\odot^K TM \otimes A^* \otimes B) \to 0$$

The map σ can be explicitly defined as follows:

$$\sigma : \Delta \longmapsto \sigma_\Delta$$

$f_i \in C^\infty(M)$

$$\sigma_\Delta(df_1, \ldots, df_K) := c_{f_1} \circ \cdots \circ c_{f_K}(\Delta)$$

The symmetry of σ_Δ follows from $c_f \circ c_g = c_g \circ c_f$, which is easily seen by construction, and the fact that it acts as a multiderivation follows from the following fact: $\Delta \in Diff_1(A,B)$, $f,g \in C^\infty(M)$, $a \in \Gamma(A)$

$$\sigma_\Delta(df)(a) := [\Delta, f](a) := \Delta(f\cdot a) - f\cdot\Delta(a)$$

$$[\Delta, fg](a) = \Delta(f\cdot(g\cdot a)) - f\cdot(g\cdot\Delta(a))$$

$$= [\Delta,f](g\cdot a) + f\cdot\Delta(g\cdot a) - f\cdot(g\cdot\Delta(a))$$

Δ is of order ≤ 1

$$= g\cdot([\Delta,f](a)) + f\cdot([\Delta,g](a))$$

So clearly $[\Delta, -](a) \in Der(C^\infty(M)) \cong \Gamma(TM)$.

II.3 Jet Bundles of Vector Bundles

A similar construction to the jet bundle of a manifold, replacing local functions by local sections mutatis mutandis, gives us the k-jet bundle of a vector bundle $\alpha: A \to M$:

$$J^k A := \bigcup_{x\in M} J^k_x A \quad , \quad J^k_x A = \{ j^k_x s,\ s \in \Gamma(A)\}.$$

And similarly, by construction, we have maps,

$$\pi^k: J^k A \to J^{k-1}A \qquad \text{and} \qquad j^k: \Gamma(A) \to \Gamma(J^k A).$$

Jet bundles of vector bundles serve a similar purpose to those of manifolds: they realize spaces of differential operators as sections of vector bundles.

Proposition 5.2 $Diff_k(A,B) \cong \Gamma((J^k A)^* \otimes B)$.

Proof. (see section 11.47 of Nestruev). //

II.4 The Notion of Locality

Proposition 5.3 Let $s, r \in \Gamma(A)$ two (possibly local) sections, $\Delta \in Diff_k(A,B)$ and $x \in U \subset M$ some neighbourhood, then:

$$s|_U = r|_U \Rightarrow \Delta(s)(x) = \Delta(r)(x).$$

proof. (see section 9.61 of Nestruev) //

This result formally encapsulates the notion that differential operators are local.

From an algebraic point of view, locality is reflected in the behaviour of the support of functions and sections:

$$\operatorname{supp}(a) = \{x \in M \mid a(x) \neq 0\}.$$

Indeed, for $f, g \in C^\infty(M)$ and $s \in \Gamma(A)$:

$$\operatorname{supp}(fg) = \operatorname{supp}(f) \cap \operatorname{supp}(g) \quad , \quad \operatorname{supp}(f \cdot s) = \operatorname{supp}(f) \cap \operatorname{supp}(s).$$

It can be seen that differential operators behave in this fashion

$$\Delta \in \mathrm{Diff}_1(A, B) \Rightarrow \quad \operatorname{supp}(\Delta(s)) = \operatorname{supp}(\Delta) \cap \operatorname{supp}(s).$$

II.5 Differential Operators of Order ≤ 1

For the reminder of these lectures, a differential operator will be understood to be of order ≤ 1.

Proposition 5.4 (Leibniz characterisation) A differential operator $\Delta \in \mathrm{Diff}_1(A, B)$ is uniquely characterised by a pair (Δ, δ) where $\Delta : \Sigma \subset \Gamma(A) \to \Gamma(B)$ is a $\mathbb{R}$-linear map and $\delta : \Gamma(T^*M) \to \Gamma(A^* \otimes B)$ is a $C^\infty(M)$-linear map such that:

spanning sections

$$\Delta(f \cdot s) = f \cdot \Delta(s) + \delta(df)(s) \qquad \forall f \in C^\infty(M),\ s \in \Sigma.$$

proof. This is a direct consequence of the symbol exact sequence and the fact that Σ is assumed to span all $\Gamma(A)$ as $C^\infty(M)$-linear combinations. //

If we now focus on $\mathrm{Diff}_1(A)$, the symbol sequence takes up the form:

$$0 \to \Gamma(\mathrm{End}(A)) \hookrightarrow \mathrm{Diff}_1(A) \xrightarrow{\sigma} \Gamma(TM \otimes \mathrm{End}(A)) \to 0$$

All these maps are, in fact, $C^\infty(M)$-linear, so we have an induced short exact sequence of vector bundles

$$0 \longrightarrow \mathrm{End}(A) \longrightarrow (J^1A)^* \otimes A \longrightarrow TM \otimes \mathrm{End}(A) \to 0$$

Taking the A-adjoint of this gives the Spencer sequence:

$$0 \longrightarrow T^*M \otimes A \xrightarrow{i} J^1A \xrightarrow{\pi^1} A \to 0$$

where the injective map i can be understood as being defined from the fact that the jet prolongation $j^1 : \Gamma(A) \to \Gamma(J^1A)$ behaves as a universal differential operator:

$$j^1(f \cdot s) = f \cdot j^1 s + i_*(df \otimes s) \qquad \forall f \in C^\infty(M),\ s \in \Gamma(A).$$

The jet prolongation is a right splitting for the Spencer sequence.

at the level of sections since it is trivial to check by construction that $\pi^1_* \circ j^1 = id_{\Gamma(A)}$, then there is an induced isomorphism of $C^\infty(M)$-modules:

$$\Gamma(J^1A) \cong \Gamma(T^*M \otimes A) \oplus_{C^\infty(M)} \Gamma(A).$$

(III) DERIVATIONS ON VECTOR BUNDLES

The vector bundle manifestation of the fact that algebras are included as multiplicative submodules in modules becomes the fact that for a vector bundle $A \to M$, there is a natural subbundle given by the injective map:

$$\mu: \mathbb{R}_M \longrightarrow End(A) \quad , \quad \mu_*: C^\infty(M) \hookrightarrow End_{C^\infty(M)}(\Gamma(A))$$

This can be seen as the presence of a globally non-vanishing section $\quad id_A \in \Gamma(End(A))$ since $\mu(\mathbb{R}_M) = C^\infty(M)\cdot id_A$

Recall the symbol sequence for vector bundles:

$$0 \to End(A) \to (J^1A)^* \otimes A \xrightarrow{\sigma} TM \otimes End(A) \to 0$$

The left inclusion tells us that id_A is (rather obviously) a differential operator (and so are all its $C^\infty(M)$-linear combinations) and the right projection (symbol map) tell us that it is natural to ask whether the symbol of a differential operator has $End(A)$ tensor factor in the span of id_A. In other words, since the symbol of a differential operator σ_Δ can be seen as a $C^\infty(M)$-derivation (or vector field) with values in $End(A)$, it is natural to ask whether the endomorphism value is the constant id_A (the only canonical non-zero choice).

Thus we arrive at the definition of derivations of a vector bundle

$$Der(A) := \sigma^{-1}(\Gamma(TM \otimes id_A)) \subset Diff_1(A).$$

Using the Leibniz characterization of differential operators we can easily unpack this definition and find the more standard presentation:

$$\mathrm{Der}(A) = \left\{ D \in \mathrm{Diff}_1(A) \,\middle|\, \exists X_D \in \Gamma(TM) : D(f\cdot s) = f\cdot D(s) + X_D[f]\cdot s \;\forall \begin{smallmatrix} f\in C^\infty(M) \\ s\in\Gamma(A)\end{smallmatrix} \right\}$$

Note that this states the following in terms of the symbol map:

$$D \in \mathrm{Der}(A) \iff \sigma_D = X_D \otimes \mathrm{id}_A .$$

Since the symbol sequence above is valid at the level of vector bundles (not only sections) with the symbol map being surjective with kernel of constant rank, derivations can be seen at the level of vector bundles as the following subbundle:

$$DA := \sigma^{-1}(TM \otimes \mathrm{id}_A) \subset (J^1A)^* \otimes A .$$

This subbundle of the bundle of differential operators is characterised by the following commutative diagram of vector bundle morphisms covering the identity:

$$\begin{array}{ccc} DA & \hookrightarrow & (J^1A)^*\otimes A \\ a\downarrow & & \downarrow\sigma \\ TM & \xrightarrow[-\otimes \mathrm{id}_A]{} & TM\otimes \mathrm{End}(A) \end{array}$$

This is the **der bundle** of the vector bundle A.

Another, more geometric, way to characterise derivations of a vector bundle using the vector bundle analogue of the parallel between vector fields and infinitesimal diffeomorphisms. An **infinitesimal automorphism** on a vector bundle $A \to M$ is a 1-parameter sub group $\{F^t : A \to A\}_{t\in I} \subset \mathrm{Aut}(A)$ covering a 1-parameter subgroup of diffeomorphisms $\{\varphi^t : M \to M\}_{t\in} \subset \mathrm{Diff}(M)$ with $(F^0, \varphi^0) = (\mathrm{id}_A, \mathrm{id}_M)$.

Proposition 5.5 An infinitesimal automorphism $\{F^t\}_{t\in I} \subset \mathrm{Aut}(A)$ defines a derivation $D^F \in \mathrm{Der}(A)$ via the following formula:

$$D^F(s) := \frac{d}{dt}\left\{ F^t_* s \right\}_{t=0} \qquad \forall\, s \in \Gamma(A).$$

proof. By the properties of push-forwards and taking tangent vectors we see that D^F is clearly $\mathbb{R}$-linear. Using a local trivialization to compute tangent vectors as derivatives on the fibres and tangent vectors on the base we can use the usual Leibniz property of derivatives to write:

$$\frac{d}{dt}\left\{(F^t)_*(f\cdot s)\right\}_{t=0} = \frac{d}{dt}\left\{(\varphi^{-t})^* f \cdot F^t_* s\right\}_{t=0} =$$

$$= f\cdot \frac{d}{dt}\left\{F^t_* s\right\}_{t=0} + \frac{d}{dt}\left\{(\varphi^{-t})^* f\right\}\cdot s$$

Now, denote $X^\varphi \in \Gamma(TM)$ the vector field associated with the infinitesimal diffeomorphism $\{\varphi^t\}_{t\in I}$ to write:

$$D^F(f\cdot s) = \frac{d}{dt}\left\{(F^t_*)(f\cdot s)\right\}_{t=0} = f\cdot D^F(s) - X^\varphi[f]\cdot s.$$

Hence, we have shown that D^F is a derivation of A covering the vector field $-X^\varphi$. //

The converse construction, i.e. integrating derivations into infinitesimal automorphisms, will be discussed in a future lecture in the context of integrability of Lie algebroids by Lie groupoids.